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Canonical Formalism and Equations of Motion for a Spinning Particle in General Relativity

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A canonical formalism for spinning particles suitable for the formulation of a general relativistic covariant statistical mechanics of particles endowed with spin is developed. For that purpose the bracket for internal and external variables is given. In particular, limiting consideration to the spin tensor $S^{\mu\nu}$, it has been possible to define momenta that are the true conjugates to the position variables. For the case considered the Hamiltonian function in addition to the invariant "mass" involves only one additional scalar. The equations of motion are then found by calculating the brackets of the dynamical variables with that Hamiltonian, and are compared to those obtained by other methods. The conjugate variables in the internal space that has to be adjoined to the (eight-dimensional) phase space for a complete covariant description of spinning particles are also given.

1. INTRODUCTION

There exists a considerable literature (Hehl *et al.*, 1976) concerning a covariant description of spinning particles. The first description was given by Frenkel (1926), while the classical work was carried out by Weyssenhoff and Raabe (1947). Following this was the work by Papapetrou (1951) and others (e.g., Nyberg, 1962; Ellis, 1966, 1970, 1971; Dixon, 1964, 1970a,b). Their approach was based on considering multiple moments, and, in particular, those for a pole–dipole particle. Later papers (Bhaba and Corben, 1941; Halbwachs, 1960; Bailey and Israel, 1975) considered a Lagrangian formalism from which the equations are derived via an action principle. More satisfactory, perhaps, is a canonical formalism.

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Introducing a generalized Hamiltonian function M and momenta p_μ defined by

$$M^2 = g^{\alpha\beta} p_\alpha p_\beta \tag{1a}$$

$$p_\mu = M\, dx_\mu / ds \tag{1b}$$

where the four canonical velocities dx_μ/ds satisfy

$$(dx_\mu/ds)(dx_\nu/ds)g^{\mu\nu} = (dx^\mu/ds)(dx^\nu/ds)g_{\mu\nu} = 1$$

the covariant canonical equations for spinless particles are given by Tauber and Weinberg (1961) and Ehlers (1961)

$$dx^\mu/ds = (\partial M/\partial p_\mu)_x \tag{2a}$$

and

$$dp_\mu/ds = -(\partial M/\partial x^\mu)_p \tag{2b}$$

as a result of which the invariant "mass" M is a constant of the motion

$$dM/ds = (\partial M/\partial x^\mu)_p\, dx^\mu/ds + (\partial M/\partial p_\mu)_x\, dp_\mu/ds = 0 \tag{3}$$

Alternately, if (3) holds, the equations of motion (Pauli, 1958) written in terms of the momenta

$$dp_\mu/ds = -(\partial M/\partial x^\mu)_p + (p_\mu/M)\, dM/ds \tag{4}$$

are equivalent to (2b), with (2a) being just a definition of p^μ/M. It is perhaps interesting to note (Tauber and Weinberg, 1961) that the set (x^μ, p_μ) transform like an eight-vector and, of course, that (2) and (3) can be written in terms of Poisson brackets as

$$dx^\mu/ds = (x^\mu, M); \qquad dp_\mu/ds = (p_\mu, M); \qquad dM/ds = (M, M) \tag{5}$$

Electromagnetic fields can be introduced by the simple device of replacing p_μ by $p_\mu + eA_\mu$ and at the same time performing a gauge transformation on the potential $A_\mu \to A_\mu + \partial G/\partial x^\mu$.

The positions x^μ and momenta p_μ then define an eight-dimensional (phase) space in which the differential volume $d^4p\, d^4x$ is an invariant. In that space one can define an invariant density-in-space $N = N(x, p)$ so that

$$(\partial/\partial x^\mu)_p(N\, dx^\mu/ds) + (\partial/\partial p_\mu)_x(N\, dp_\mu/ds) = 0 \tag{6}$$

which, due to (2), reduces to

$$\begin{aligned} dN/ds &= (\partial N/\partial x^\mu)_p\, dx^\mu/ds + (\partial N/\partial p_\mu)_x\, dp_\mu/ds \\ &= \mathrm{Div}(N\, dx/ds, N\, dp/ds) = (N, M) = 0 \end{aligned} \tag{7}$$

which is the covariant form of Liouville's theorem.

Quantities of physical interest, such as the current vector density, energy momentum density, and entropy density, are then obtained by integrating N multiplied by appropriate factors p_μ over momentum space. In general, if $K = K(N, M)$, it follows from (7) that the covariant derivative of

$$\mathcal{K}_{\mu\nu\lambda\cdots} = \int K(N, M)\, d^4p\, p_\mu p_\nu p_\lambda \cdots \qquad (8)$$

is identical to zero (Tauber and Weinberg, 1961; Israel, 1963), i.e.,

$$K^{\mu}_{\nu\lambda\cdots;\mu} = 0$$

In the case of spinning particles the situation is more complicated. In the first place brackets between momenta do not vanish, but involve the spin tensor $S^{\nu\nu}$ and the Riemannian curvature tensor. Also, since the mass M is no longer a constant of the motion, the simple canonical equations (2) or their equivalents do not hold. One can get around the first difficulty by defining new momenta p^*_μ, which differ from p_μ by terms involving $S^{\mu\nu}$ (in analogy to adding a vector potential) and are the true conjugates to x^μ. The relevant bracket algebra is described in Section 2. Since we can limit ourselves to the simple group $SL(c, 2)$ (for which the two scalars are constructed from $S^{\mu\nu}$ and its dual constants), it is possible to define a general Hamiltonian function H, which, in addition to the "mass" M (expressed in terms of the new momenta), involves only one additional scalar. The equations of motions are then simply obtained by calculating the brackets of the dynamical variables with the Hamiltonian. This is done in Section 3, where our results are also compared with those obtained by other methods.

Since we hope to apply our results to the formulation of a covariant statistical mechanics of particles involving spin in a future paper, we add two appendices, one in which we write down the equations for motion for two special cases suitable for further work, and one in which we obtain the variables for the internal phase space that have to be adjoined to those of the ordinary phase space.

2. THE INTERNAL BRACKET ALGEBRA[3]

The particle is characterized by the space-time variables x^μ, the momenta p_μ, and internal variables denoted generically by $Q^{\mu\cdots}$, which include at least the antisymmetric spin tensor $S^{\mu\nu}$ generating a local homogeneous Lorentz group.[4]

Virtual translations in x^μ are generated according to

$$(p_\mu, x^\nu) = \delta^\nu_\mu \qquad (10)$$

[3]This section is based on a Ph.D. thesis by Y. Feldman, Tel Aviv University (unpublished).
[4]Our discussion here is quite general, but subsequently we limit ourselves to the spin tensor $S^{\mu\nu}$.

while the independence of internal and external coordinates is formulated as

$$(x^\mu, Q^{\nu\cdots}) = 0 \tag{11}$$

The internal spin symmetry is characterized by

$$(S^{\mu\nu}, Q^{\lambda\cdots}) = g^{\nu\lambda} Q^{\mu\cdots} - g^{\mu\lambda} Q^{\nu\cdots} - \cdots \tag{12}$$

which for the particular cases of a vector and the spin tensor reduces to

$$(S^{\mu\nu}, Q^\lambda) = g^{\nu\lambda} Q^\mu - g^{\mu\lambda} Q^\nu \tag{12a}$$

$$(S^\mu_\nu, S^\alpha_\beta) = g^\alpha_\nu S^\mu_\beta - g_{\nu\beta} S^{\mu\alpha} - g^\mu_\beta S^\alpha_\nu + g^{\mu\alpha} S_{\beta\nu} \tag{12b}$$

The requirement that virtual translation subjects every internal tensor to parallel displacement is expressed by the vanishing of the covariant bracket of momentum with any internal tensor

$$(p_\mu, Q^{\nu\cdots}) + \Gamma^\nu_{\mu\lambda} Q^{\lambda\cdots} + \cdots = 0 \tag{13}$$

which suggests that p_μ are the gauge-invariant displacement generators, so that the bracket (p_μ, p_ν) can only vanish if space is flat or spin absent. Neglecting electromagnetic forces, it is given by

$$(p_\mu, p_\nu) = \tfrac{1}{2} S^\alpha_\beta R^\beta_{\alpha\mu\nu} \tag{14}$$

This, then, constitutes the underlying bracket algebra. However, internal and external variables are here intertwined, while to serve as coordinates of a phase space they must be disentangled, unless they are canonically conjugate to each other. In particular, we require the bracket of p_μ with p_ν, (14), or with Q^ν to vanish, while maintaining (10) and (11).

We can achieve this by making use of the isomorphism between the internal gauge group and the geometrical symmetry group of the tangent space and replace the gravitational potentials by tetrad fields

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu \tag{15}$$

where $\eta_{ab} = \mathrm{diag}(-1, -1, -1, 1)$. The extra (six) fields are exactly what is needed to admit a six-parameter internal gauge group. The indices a, b are sensitive to internal Lorentz transformations alone and do not respond to coordinate transformations. Raising and lowering of indices is carried out in the usual manner by

$$h^a_\mu = \eta^{ab} h_{b\mu} \qquad \text{and} \qquad h^\mu_a = g^{\mu\nu} h_{a\nu} \tag{15a}$$

Due to the Jacobi identities, it will be sufficient to reduce (13) in order to disentangle the momenta. It can be shown that writing

$$p^*_\mu = p_\mu - \tfrac{1}{2} h^a_{\lambda;\mu} h_{a\nu} S^{\lambda\nu} \tag{16}$$

makes p^*_μ the true canonical conjugate to x^m,

$$(p^*_\mu, x^\nu) = \delta^\nu_\mu; \qquad (p^*_\mu, Q^\alpha) = (p^*_\mu, p^*_\nu) = 0 \tag{17}$$

while the remaining brackets become

$$(Q^\lambda, x^\mu) = 0 \tag{17a}$$

$$(S^{ab}, Q^c) = \eta^{bc} Q^a - \eta^{ac} Q^b \tag{17b}$$

where

$$Q^a = h^a_\mu Q^\mu \qquad \text{and} \qquad S^{ab} = h^a_\mu h^b_\nu S^{\mu\nu}$$

In the case described here the internal structure of the particle is described by a spin tensor $S^{\mu\nu}$ (or rather S^{ab}) and four internal external vectors Q^μ_A ($A = \text{I, II, III, IV}$) satisfying (12). Associated with S^{ab} and its dual

$${}^*S^{ab} = \tfrac{1}{2} e^{abcd} S_{cd}$$

are two real scalars s and s^* constructed from $S^2 = S^a_b S^b_a$ and ${}^*S^2$

$$S^2 - {}^*S^2 = s^2 - s^{*2}; \qquad {}^*SS = ss^* \tag{18}$$

related to the proper values of the matrix S. The vectors Q^μ_A can be used to construct a spin-aligned tetrad, satisfying the completeness and orthogonality conditions

$$Q^\mu_A Q^\nu_B \eta^{AB} = g^{\mu\nu} \qquad \text{and} \qquad g_{\mu\nu} Q^\mu_A Q^\nu_B = \eta_{AB} \tag{19}$$

It can then be shown that $S^{\mu\nu}$ can be expressed in terms of the two scalars s and s^* and the vectors Q^μ_A

$$S^{\mu\nu} = s(Q^\mu_{\text{III}} Q^\nu_{\text{IV}} - Q^\nu_{\text{III}} Q^\mu_{\text{IV}}) - s^*(Q^\mu_{\text{I}} Q^\nu_{\text{II}} - Q^\nu_{\text{I}} Q^\mu_{\text{II}}) \tag{20}$$

The introduction of two additional antisymmetric tensors formed by cyclic permutation of I, II, and III in (20) then closes the Lie algebra. However, for the special case that we wish to consider

$$s = \text{const}, \qquad s^* = \text{const} \tag{21}$$

the decomposition indicated by (20) is not possible and we arrive at the minimal structure consisting of spin alone, subject to (21).

3. EQUATIONS OF MOTION

For constant s and s^*, (21), the only independent scalars that can be formed are

$$M = (p^*_\mu p^{*\mu})^{1/2}, \qquad L = (p^*_\alpha p^*_\beta \, {}^*S^a_c \, {}^*S^{cb} h^\alpha_a h^\beta_b)^{1/2} \tag{22}$$

so that the Hamiltonian is

$$H = H(M, L) \tag{23}$$

For any function f of the dynamical variables we have

$$df/d\tau = (H, f) = (\partial H/\partial M)(M, f) + (\partial H/\partial L)(L, f) \tag{24}$$

where τ is a parameter monotonic in proper time, but not necessarily the proper time itself. Taking f in turn to be x^μ, p^*_μ, and $*S^{ab}$, we obtain

$$dx^\mu/d\tau = M^{-1}(\partial H/\partial M)p^{*\mu} + L^{-1}(\partial H/\partial L)\, {*S^a_b}\, {*S^{bc}} h^\mu_a h^\nu_c p^*_\nu \tag{25a}$$

$$\begin{aligned} Dp^*_\mu/D\tau &\equiv dp^*_\mu/d\tau - \Gamma^\nu_{\mu\lambda} p^*_\nu \, dx^\lambda/d\tau \\ &= -p_\nu (dx^\lambda/d\tau) h^a_\lambda h^\nu_{a;\mu} \end{aligned} \tag{25b}$$

$$dS^{ab}/d\tau + v^a p^{*b} - v^b p^{*a} = 0 \tag{25c}$$

where we defined $v^a = (dx^\mu/d\tau)h^a_\mu$. In order that τ be the proper time, we have to satisfy the equation

$$(dx^\mu/d\tau)(dx^\nu/d\tau)g_{\mu\nu} = 1 \tag{26}$$

Making use of (15), we find

$$v^a v^b \eta_{ab} = 1 \tag{26c}$$

The energy E, which we would like to be a constant, is given by

$$E = p^*_\mu \, dx^\mu/d\tau = (\partial H/\partial M)M + (\partial H/\partial L)L \tag{27}$$

Thus, if H is a homogeneous function of M and L, E will be a constant of the motion.

So far we have assumed that all the x dependence enters only through the metric, and the electromagnetic interaction only through the momenta. We could, of course, also add nonminimal couplings, such as $gS^{ab}F_{\mu\nu}h^\mu_a h^\nu_b$ or $S^{\mu\nu}S^{\alpha\beta}R_{\mu\nu\alpha\beta}$, etc. Since x^μ has vanishing brackets with the additional term, its equation of motion (25a) will not change. The equation for p^*_μ, (25b), will acquire the additional terms

$$gS^{ab}F_{\nu\alpha,\mu}h^\nu_a h^\alpha_b + gS^{ab}F_{\nu\alpha}(h^\nu_a h^\alpha_b)_{,\mu} \tag{25b$'$}$$

where the first term gives the interaction of the dipole with the gradient of the electromagnetic field and the second term is the contribution of curved space-time; g is the coupling constant. The equation of motion for the spin (25c) will acquire the additional terms

$$gF_{\mu\nu}[S^{ab}(h^{b\mu}h^\nu_c - h^\mu_c h^{b\nu}) + S^{bc}(h^\mu_c h^{a\nu} - h^{a\mu}h^\nu_c)] \tag{25c$'$}$$

giving the action of a local torque on the spin.

It might also be instructive to compare our results with those obtained by others. Introducing again p_μ (instead of p^*_μ), we obtain from (25b), (25b′), (14), and (16)

$$Dp_\mu/D\tau = -eF_{\mu\nu}v^\nu - \tfrac{1}{2}S^\alpha_\beta R^\beta_{\alpha\mu\nu}v^\nu + gS^{\nu\lambda}F_{\nu\lambda,\mu} + gS^{ab}F_{\nu\lambda}(h^\nu_a h^\lambda_b)_{,\mu} \quad (25\text{b}'')$$

We note that Bargmann *et al.* (1959) and Papapetrou (1951), who only considered minimal couplings, only have the first term on the right-hand side of (25b″). On the other hand, Corben (1968), Frenkel (1926), Weyssenhoff and Raabe (1947), and Halbwachs (1960) have equations identical to (25b″) with momenta defined by

$$p_\mu = mv_\mu + \dot{S}_{\mu\nu}v^\nu + gF_{\nu\lambda}S^\lambda_\mu v^\nu$$

Ellis (1970, 1971), using a different notation, arrives at the same results.

The spin equation (25c), (25c′) can be written in the form

$$dS^{ab}/d\tau + v^a v^b - v^b v^a = 2g(S^b_c F^{ca} - S^a_c F^{cb}) \quad (25\text{c}'')$$

where we *disregarded* terms arising from the curvature of space-time not considered by others. This equation is identical to those obtained by the authors cited above, who use the equivalent form

$$\dot{S}_{\mu\nu} - v^\lambda(\dot{S}_{\mu\lambda}v_\nu - \dot{S}_{\nu\lambda}v_\mu) = F_{\nu\lambda}m^\lambda_\mu - F_{\mu\lambda}m^\lambda_\nu + v^\alpha F_{\alpha\beta}(m^\beta_\mu v_\nu - m^\beta_\nu v_\mu)$$

where $m^{\mu\nu} = gS^{\mu\nu}$.

Finally, we would like to comment on a striking similarity between this work and that of Ellis (1970, 1971) and Halbwachs (1960). They assume the electromagnetic dipole to be of the form

$$p^{\mu\nu} = q^\mu v^\nu - q^\nu v^\mu + \delta^{\mu\nu\alpha\beta}m_\alpha v_\beta \quad (28)$$

If one now defines

$$q = (-q^\mu q_\mu)^{1/2}, \qquad m = (-m^\nu m_\nu)^{1/2},$$
$$q^\mu = qs^\mu,\ m^\nu = ms^\nu \qquad (s_\mu s^\mu = -1)$$

the above expression can be put in the form

$$p^{\mu\nu} = q(s^\mu s^\nu - s^\nu s^\mu) - \tfrac{1}{2}m\delta^{\mu\nu\alpha\beta}(s_\alpha v_\beta - s_\beta v_\alpha) \quad (28\text{a})$$

where clearly the second term is the adjoint of the first, and one has a decomposition similar to (20).

APPENDIX A. EQUATIONS OF MOTION FOR PARTICULAR CHOICES OF THE TETRAD

In deriving the equations of motion, the tetrad fields have been introduced as an arbitrary frame of reference, but in order to use these equations of motion, the frame has to be specified.

Since there are two timelike vectors $dx^\mu/d\tau$ and p^*_μ, one has two "natural" possibilities for the timelike h^μ_4. Taking h^μ_4 parallel to the momentum means $h^\mu_4 = M^{-1}p^*_\mu$, which implies $p^* = p^*_\mu h^{\mu a} = \delta^a_4$, so that the equations of motion become

$$dx^4/d\tau = \partial H/\partial M + (L/M)\,\partial H/\partial L = E/M \tag{A1a}$$

$$dx^a/d\tau = (M/L)(\partial H/\partial L)\,{}^*S^a_b\,{}^*S^{b4} \qquad (a \neq 4) \tag{A1b}$$

$$dp^*_4/d\tau = dM/d\tau = (dx^\lambda/d\tau)h_{4\lambda;\mu}p^{*\mu} + p^*_\mu\, Dh^\mu_4/D_\tau \tag{A1c}$$

$$dp^*_a/d\tau = (dx^\lambda/d\tau)h^4_{\lambda;\mu}h^\mu_a M + p^*_\mu\, Dh^\mu_a/D\tau \qquad (a \neq 4) \tag{A1d}$$

$$dS^{ab}/d\tau + (v^a\delta^b_4 - v^b\delta^a_4)M = 0 \tag{A1e}$$

The second possibility is $h^\mu_4 = dx^\mu/d\tau$ and then $v^a = \delta^a_4$, while the equations of motions become

$$dx^4/d\tau = 1 \tag{A2a}$$

$$dx^a/d\tau = 0 \qquad (a \neq 4) \tag{A2b}$$

$$dp^*_4/d\tau = 0 \tag{A2c}$$

$$dp^*_a/d\tau = (dx^\nu/dt)_{;\mu}(-p^*_\nu h^\mu_a - p^{*\mu}h_{a\nu}) \qquad (a \neq 4) \tag{A2d}$$

$$dS^{ab}/d\tau + (\delta^a_4 p^*_b - \delta^b_4 p^{*a}) = 0 \tag{A2e}$$

In this case $E = p^*_4$, making it automatically a constant.

APPENDIX B. GENERALIZED PHASE SPACE OF A SPINNING PARTICLE

The bracket algebra considered in Section 2 is not a linear representation of a Lie algebra, but rather a function group. There exists then an underlying manifold that spans the algebra. If the number of dimensions of that manifold equals the number of functionally independent generators, as is the case here, all the Casimir operators vanish identically.

The functionally independent generators of the algebra can be grouped into two sets y^A and z_A, which satisfy the commutation relations[5]

$$(y^A, y^B) = 0, \qquad (z_A, z_B) = 0, \qquad (y^A, z_B) = \delta^A_B \tag{B1}$$

Clearly, it is those variables that will become the canonical variables of the internal phase space. By a method similar to the one used by Kramers (1935) and Schiller (1962), who considered the generators of $O(3)$,

$$A_1 = S^{23}, \qquad A_2 = S^{31}, \qquad A_3 = S^{12} \qquad \text{with } \sum A_i^2 = \text{const} \tag{B2}$$

[5]In our case $A, B = 1, 2$.

for which the two independent canonical variables are

$$z = A_3 \qquad \text{and} \qquad y = \tan^{-1}(A_1/A_2) \tag{B3}$$

we can decompose the $SL(2, c)$ algebra into a direct product $O(3) \times O(3)$ by defining

$$C_j = A_j + iB_j, \qquad D_j = A_j - iB_j \qquad (j = 1, 2, 3) \tag{B4}$$

where

$$\begin{aligned} A_1 &= S^{23}, & A_2 &= S^{31}, & A_3 &= S^{12} \\ B_1 &= S^{14}, & B_2 &= S^{24}, & B_3 &= S^{34} \end{aligned}$$

for which the brackets are given by

$$(C_i, C_j) = e_{ijk} C_k, \qquad (D_i, D_j) = e_{ijk} D_k, \qquad (C_i, D_j) = 0 \tag{B5}$$

with

$$\sum C_i^2 = (s + is^*)^2 = c^2 \qquad \text{and} \qquad \sum D_i^2 = (s - is^*)^2 = d^2$$

Since the brackets of s and s^* with all other variables vanish, they are constants of the motion, as we postulated previously. In analogy with (B3), we find that the canonical variables can be written as

$$\begin{aligned} z_1 &= C_3, & y^1 &= \tan^{-1}(C_1/C_2) \\ z_2 &= D_3, & y^2 &= \tan^{-1}(D_1/D_2) \end{aligned} \tag{B6}$$

and obey conditions

$$y^{1*} = y^2, \qquad z_1^* = z_2 \tag{B6a}$$

In terms of y^A and z_A

$$\begin{aligned} S^{12} &= \tfrac{1}{2}(z_1 + z_2) \\ S^{34} &= \tfrac{1}{2}(z_1 - z_2) \\ S^{31} &= \tfrac{1}{2}(X \cos y^1 + Y \cos y^2) \\ S^{23} &= \tfrac{1}{2}(X \sin y^1 + Y \sin y^2) \\ S^{14} &= (2i)^{-1}(X \sin y^1 - Y \sin y^2) \\ S^{24} &= (2i)^{-1}(X \cos y^1 - Y \cos y^2) \end{aligned} \tag{B7}$$

where

$$X = (c^2 - z_1^2)^{1/2}, \qquad Y = (d^2 - z_2^2)^{1/2}$$

The volume element in the internal phase space, which has to be adjoined to the one in ordinary phase space $d^4x\, d^4p^*$, is given by

$$d^4 V = cd\, dz_1\, dz_2\, dy^1\, dy^2 \tag{B8}$$

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